Measure Theory

The problems below are taken out of various textbooks on real variables, including "Real Analysis" by Elias M. Stein and Rami Shakarchi and "Real Analysis" by N. L. Carothers. Questions are also taken from real variables qualifying exams at CUNY Graduate Center. The problems are color-coded. The color green indicates that the problem came from a textbook and to the best of my knowledge was not featured on any qualifying exam. Yellow means that the problem was spotted in at least one qualifying exam. Red indicates that the problem or one just like it appeared in at least two qualifying exams.

1. Let X be a compact metric space and let $\mathcal{O}$ be an open cover of X. Show there exists a real number $\epsilon > 0$ such that any ball of radius less than ϵ is contained in at least one of the sets comprising $\mathcal{O}$

2. Suppose E is a given set and $\mathcal{O}_n$ is the open set:
$$\mathcal{O}_n = \{x : d(x, E) < 1/n\}$$
Show:
   (a) If E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$
   (b) However, the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

3. Using translations and dilations, prove the following: Let B be a ball in $\mathbf{R}^d$ of radius r. Then $m(B) = v_d r^d$, where $v_d = m(B_1)$, and $B_1 = \{x \in \mathbf{R}^d : |x| < 1\}$.

4. If $\delta = (\delta_1,...,\delta_d)$ is a d-tuple of positive numbers $\delta_i > 0$, and E is a subset of $\mathbf{R}^d$, we define δE by
$$\delta E = \{(\delta_1 x_1,...,\delta_d x_d): \text{where } (x_1,...,x_d) \in E\}.$$
Prove that δE is measurable whenever E is measurable, and
$$m(\delta E) = \delta_1...\delta_d m(E).$$

5. Suppose L is a linear transformation of $\mathbf{R}^d$. Show that if E is a measurable subset of $\mathbf{R}^d$, then so is L(E), by proceeding as follows:
   (a) Note that if E is compact, so is L(E). Hence if E is an F_σ set, so is L(E).
   (b) Because L automatically satisfies the inequality
$$|L(x) - L(y)| \le M|x - y|$$
   for some M, we can see that L maps any cube of side length ℓ into a cube of side length $c_d M \ell$, with $c_d = 2\sqrt{d}$. Now if m(E) = 0, there is a collection of cubes $\{Q_j\}$

such that $E \subset \bigcup_{j=1}^{\infty} Q_j$, and $\sum_{j=1}^{\infty} m(Q_j) < \varepsilon$. Thus $m^*(L(E)) \le c' \varepsilon$, and hence $m(L(E)) =$ 0. Finally observe that $L(E)$ is measurable if and only if $L(E)$ is the union of an F_σ set and a set of measure 0.

6. Give an example of an open set $\mathcal{O}$ with the following property: the boundary of the closure of $\mathcal{O}$ has positive Lebesgue measure.

7. Let A be the subset of $[0, 1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$.

8. The following deals with G_δ and F_σ sets.
   (a) Show that a closed set is a G_δ and an open set an F_σ .
   (b) Give an example of an F_σ which is not a G_δ.
   (c) Give an example of a Borel set which is not a G_δ nor an F_σ .

9. The **outer Jordan content** $J^*(E)$ of a set E in **R** is defined by
$$J^*(E) = \inf \sum_{j=1}^{N} I_j ,$$
where the inf is taken over every *finite* covering $E \subset \bigcup_{j=1}^{N} I_j$.
   (a) Prove that $J^*(E) = J^*(\overline{E})$ for every set E (here $\overline{E}$ denotes the closure of E).
   (b) Exhibit a countable subset $E \subset [0, 1]$ such that $J^*(E) = 1$ while $m^*(E) = 0$.

10. **The Borel-Cantelli lemma.** Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that
$$\sum_{k=1}^{\infty} m(E_k) < \infty$$
Let
$$E = \{x \in \mathbf{R}^d : x \in E_k , \text{ for infinitely many } k\} = \limsup_{k \to \infty}(E_k).$$
   (a) Show that E is measurable.
   (b) Prove $m(E) = 0$

11. Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ with $| f_n | < \infty$ for a.e. x. Show that there exists a sequence c_n of positive real numbers such that
$$\frac{f_n(x)}{c_n} \to 0 \text{ a.e. x}$$

12. Suppose $\{f_n\}$ is a sequence of measurable functions on $[0, 1]$ which converges to zero almost everywhere. Prove that there exists a sequence $\{t_n\}$ of real numbers such that $\sum_{n=1}^{\infty} |t_n f_n(x)| < \infty$ a.e. $x \in [0, 1]$

13. Prove the following assertion: Every measurable function is the limit a. e. of a sequence of continuous functions.

14. Here are some observations regarding the set operation A + B.
   (a) Show that if either A or B is open, then A + B is open.
   (b) Show that if A and B are closed, then A + B is measurable.
   (c) Show, however, that A + B might not be closed even though A and B are closed.

15. Show that there exist closed sets A and B with $m(A) = m(B) = 0$, but $m(A + B) > 0$:
   (a) In $\mathbf{R}$, let $A = \Delta$ (the Cantor set), $B = \Delta/2$. Note that $A + B \supset [0, 1]$
   (b) In $\mathbf{R}^2$, observe that if $A = [0, 1] \times \{0\}$ and $B = \{0\} \times [0, 1]$, then $A + B = [0, 1] \times [0, 1]$

16. Let $A \subset \mathbf{R}$. If $m^*(A) > 0$, show that A contains a nonmeasurable set.

17. Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set.

18. Show that there are measurable sets that are not Borel sets.

19. Does there exist an enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals, such that the complement of

$$\bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n} \right)$$

in $\mathbf{R}$ is non-empty?

20. Suppose $A \subset E \subset B$, where A and B are measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is measurable.

21. Suppose $A \subset \mathbf{R}$ is measurable with $m(A) < \infty$. Show that for every number $x \in (0, m(A))$, there exists a compact set $F \subset A$ containing no rational numbers such that $m(F) = x$.

22. Suppose that $E \subset \mathbf{R}$ and $m^*(E) > 0$. Given $0 < \alpha < 1$, show that there exists an open interval I such that $m^*(E \cap I) > \alpha \, m(I)$.

23. Let $E \subset \mathbf{R}$ be measurable with m(E) > 0. Prove that $E - E = \{x - y : x, y \in E\}$ contains an interval centered at 0.

24. If E and F are measurable, and m(E) > 0, m(F) > 0, prove that $E + F = \{x + y : x \in E, y \in F\}$ contains an interval.

25. Let $E \subset \mathbf{R}$ be Lebesgue measurable, and let $r \in \mathbf{R}$. The Lebesgue density of E at r is defined to be

$$\lim_{h \downarrow 0} \frac{m(E \cap [r - h, \ r + h]}{2h}$$

provided the limit exists. Construct a measurable subset $E \subset \mathbf{R}$ for which the density does not exist for at least one $r \in \mathbf{R}$.

Solutions:

1. Suppose that no such number ϵ exists for the open cover $\mathcal{O}$ of X. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that the open ball of radius 1/n centered at x_n, $B_{1/n}(x_n)$, is not contained in any member U_λ of the collection $\mathcal{O}$. Since X is compact, any sequence of its elements has a convergent subsequence. Let $\{x_{n(k)}\}_{k=1}^{\infty}$ be a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ which converges to some element $x \in X$. This x is contained in some member U_λ of the open cover $\mathcal{O}$ and therefore there is some $\delta > 0$ such that $B_\delta(x) \subset U_\lambda$. Letting k be large enough so that $1/n(k) < \delta/2$ and $d(x_{n(k)}, \ x) < \delta/2$, we obtain $B_{1/n(k)}(x_{n(k)}) \subset B_\delta(x) \subset U_\lambda$, which contradicts the choice of the sequence $\{x_n\}_{n=1}^{\infty}$.

2. (a) We will show that for any $\epsilon > 0$, $m(E) < m(\mathcal{O}_n) < m(E) + \epsilon$ for all sufficiently large n. Since E is compact, then, in particular, E is bounded and $m(E) < \infty$. Therefore there exists an open set $U \supset E$ such that $m(U) < m(E) + \epsilon$.

We can write

$$U = \bigcup_{y \in U} B_{r(y)}(y)$$

where r(y) is small enough so that $B_{r(y)}(y) \subset U$.

Observe that the collection $\{B_{r(y)}(y) : y \in U\}$ is an open cover of E. Consequently, there is some $\delta > 0$ such that for any $x \in E$, there is a $y \in U$ for which $B_\delta(x) \subset B_{r(y)}(y)$. For all n large enough so that $1/n < \delta$, $z \in \mathcal{O}_n \Rightarrow d(z, x) < 1/n$ for some $x \in E \Rightarrow z \in B_{1/n}(x) \subset B_\delta(x) \subset U$. Consequently, $\mathcal{O}_n \subset U$, which implies

$$m(E) < m(\mathcal{O}_n) < m(U) < m(E) + \epsilon$$

as desired.

(b) Consider E = $\left\{ \sum_{k=1}^{N} \frac{1}{k} : \ N \geq 1 \right\}$. Then E is an unbounded closed set consisting of isolated points. Since E is countable, m(E) = 0.

For any n > 0 and x $\geq \sum_{k=1}^{n} \frac{1}{k}$,

$$x \in \left[\sum_{k=1}^{N} \frac{1}{k}, \ \sum_{k=1}^{N+1} \frac{1}{k} \right]$$

for some N $\geq$ n.

Thus

$$d(x, E) < 1/(N+1) < 1/n,$$

and therefore

$$\left[\sum_{k=1}^{N} \frac{1}{k}, \ \infty \right) \subset \mathcal{O}_n .$$

It follows that m($\mathcal{O}_n$) = ∞

Now consider E = $\bigcup_{k=1}^{\infty} \left(r_k - \frac{\varepsilon}{2^{k+1}}, \ r_k + \frac{\varepsilon}{2^{k+1}} \right)$, where the r_k are enumerating the set [0, 1] $\cap$ **Q** and $\varepsilon < 1$. Then E is a bounded open set with m(E) $\leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$. Notice that E is dense in [0, 1]. Therefore, [0, 1] $\subset \mathcal{O}_n \Rightarrow$ m($\mathcal{O}_n$) ≥ 1

3. Let B = {x $\in$ **R**^{d} : |x - a| < r}. Then B = {a + rx: x $\in$ B_1} = a + r B_1. Thus B is just B_1 dilated by the scalar r and translated by the vector a. It follows that m(B) = m(a + r B_1) = m(r B_1) = r^d m(B_1).

4. First observe that for any rectangle R, m(δR) = $\delta_1 ... \delta_d$ m(R). For a set E of finite outer measure m*(E) and ϵ > 0 we can find an open set $\mathcal{O} \supset$ E such that m($\mathcal{O}$) < m*(E) + ϵ. The set $\mathcal{O} = \bigcup_{j=1}^{\infty} Q_j$ is a countable union of almost disjoint cubes. Therefore $\delta \mathcal{O} = \bigcup_{j=1}^{\infty} \delta Q_j$ is a countable union of almost disjoint rectangles whereby

$$m(\delta \mathcal{O}) = \sum_{j=1}^{\infty} m(\delta Q_j) = \sum_{j=1}^{\infty} \delta_1 ... \delta_d m(Q_j) = \delta_1 ... \delta_d m(\mathcal{O}).$$

Since $\delta \mathcal{O} \supset \delta$E, we have

$$m*(\delta E) \leq m(\delta \mathcal{O}) = \delta_1 ... \delta_d m(\mathcal{O}) < \delta_1 ... \delta_d m*(E) + \delta_1 ... \delta_d \ \epsilon.$$

As ϵ is arbitrary, it follows that

$$m^*(\delta E) \le \delta_1 \ldots \delta_d \, m^*(E). \tag{1}$$

Notice that E = $\delta^{-1} \delta$ E, where $\delta^{-1} = (\delta_1^{-1}, \ldots, \delta_d^{-1})$. Hence, by the inequality (1),

$$m^*(E) = m^*(\delta^{-1} \delta E) \le \delta_1^{-1} \ldots \delta_d^{-1} m^*(\delta E) \tag{2}$$

It follows by inequality (2) that

$$\delta_1 \ldots \delta_d \, m^*(E) \le m^*(\delta E). \tag{3}$$

Inequalities (1) and (3) imply the desired equality.

Finally, to show that δE is measurable whenever E is, observe that if $\mathcal{O} \supset E$ is open and $m^*(\mathcal{O} - E) < \epsilon$, then $\delta \mathcal{O} \supset \delta E$ is open and satisfies

$$m^*(\delta \mathcal{O} - \delta E) = m^*(\delta [\mathcal{O} - E]) = \delta_1 \ldots \delta_d \, m^*(\mathcal{O} - E) < \delta_1 \ldots \delta_d \epsilon.$$

5. (a) Note that any linear map L on $\mathbf{R}^d$ is Lipschitz and therefore continuous. It follows that for any compact set K, the image of K under L, L(K), is compact. By letting Q_j be any increasing sequence of closed cubes such that $\mathbf{R}^d = \lim_{j \to \infty} Q_j$, observe that every closed subset F of $\mathbf{R}^d$ is a countable union of compact sets. That is F = $\bigcup_{j=1}^{\infty} F \cap Q_j$.

Hence any F_σ set is a countable union of compact sets; If H = $\bigcup_{j=1}^{\infty} F_j$ is the union of closed sets F_j, then H = $\bigcup_{n=1}^{\infty} K_n$ is the union of compact sets K_n. Therefore L(H) = $\bigcup_{n=1}^{\infty} L(K_n)$ is the union of compact sets as well, making L(H) an F_σ set.

(b) Let E be a set of measure 0. Then, for $\epsilon > 0$, there is a sequence of closed cubes Q_j such that E $\subset \bigcup_{j=1}^{\infty} Q_j$ and m(E) $\le \sum_{j=1}^{\infty} m(Q_j) < \epsilon$. We have that L(E) $\subset \bigcup_{j=1}^{\infty} L(Q_j) \subset \bigcup_{j=1}^{\infty} \tilde{Q}_j$, where $\tilde{Q}_j$ is a cube of side length $2\sqrt{d} \, M \ell_j$ and ℓ_j is the side length of Q_j (The choice of $\tilde{Q}_j$ is dictated by the fact that for any L(x) and L(y) in L(Q_j), $|L(x) - L(y)| \le M \, |x - y| \le$ M $\sqrt{(\ell_j)^2 + \ldots + (\ell_j)^2}$ = M $\sqrt{d(\ell_j)^2}$ = $\sqrt{d}$ M $\ell_j < 2\sqrt{d}$ M ℓ_j). Thus $m^*(L(E)) \le$ $\sum_{j=1}^{\infty} m(\tilde{Q}_j) < (2\sqrt{d} \, M)^d \epsilon$. And since $\epsilon > 0$ is arbitrary, it follows that $m^*(L(E)) = 0$.

Finally, if E is a measurable subset of $\mathbf{R}^d$, then E can be expressed as a union of an F_σ set with a set of measure 0. By the above work, the image of this union under the linear map L is a union of an F_σ set with a set of measure 0. Hence every linear operator on $\mathbf{R}^d$ maps measurable sets to measurable sets.

6. For $0 < \alpha < 1$, let Δ_α be a Cantor-like set obtained as follows:

Step 1: Remove the middle open interval of length $\alpha/3$ from [0, 1]

Step 2: Remove the middle open intervals of length $(\alpha/3)^2$ from each of the 2 surviving closed subintervals of [0, 1] after step 1.

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Step n: Remove the middle open intervals of length $(\alpha/3)^n$ from each of the 2^{n-1} surviving closed subintervals of [0, 1] after step n – 1.

Continuing in this fashion, we obtain the open set G = $\bigcup\limits_{n=1}^{\infty} J_n$ where $J_n = \bigcup\limits_{k=1}^{2^{n-1}} I_k$ is the union of the 2^{n-1} middle open subintervals that were removed at step n. Hence

$$m(G) = \sum_{n=1}^{\infty} m(J_n) = \sum_{n=1}^{\infty} \frac{2^{n-1}\alpha^n}{3^n} = \frac{\alpha}{3}\frac{1}{1-(2\alpha/3)} = \frac{\alpha}{3-2\alpha}.$$

Thus, if we set $\alpha = \frac{1}{2}$, for instance, we get m(G) = $\frac{1}{4}$ and therefore m(Δ_α) = $\frac{3}{4}$ > 0.

Set $\mathcal{O} = \bigcup\limits_{n=1}^{\infty} J_{2n-1}$. Then the closure of $\mathcal{O}$ contains at least one boundary point of each interval obtained in steps 1, 2, …, n, etc. Since every point in Δ_α is a limit point of the boundary points of these intervals, the closure of $\mathcal{O}$, cl($\mathcal{O}$), must contain the entire set Δ_α. Similar analysis suggests that each point in Δ_α is also a limit point of $\mathcal{O}' = \bigcup\limits_{n=1}^{\infty} J_{2n}$.

We conclude that the boundary of the closure of $\mathcal{O}$, bdry(cl($\mathcal{O}$)), contains Δ_α as a subset and therefore 0 < m(Δ_α) ≤ m[bdry(cl($\mathcal{O}$))].

7. The set A can be constructed as follows:
    Step 1: Subdivide the interval [0, 1] into 10 equal subintervals [0, 1/10], [1/10, 2/10], etc, and remove the interval [4/10, 5/10].
    Step 2: Subdivide each of the 9 remaining subintervals further into 10 subintervals and remove all 9 intervals of the form [(10k+4)/100, (10k +5)/100] where k = 0, 1, …, 9, but does not take the value 4.

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    Step n: From each surviving subinterval from step n−1, remove 9^{n-1} subintervals of the form $\left[\sum_{j=1}^{n-1}\frac{10^j k_j + 4}{10^n}, \ \sum_{j=1}^{n-1}\frac{10^j k_j + 5}{10^n}\right]$ where each k_j = 0, 1, …, 9, but does not take the value 4.
Continuing in this fashion, we see that the measure of the union of the sets removed in steps 1, 2, …, n, etc. is the sum $\sum\limits_{n=1}^{\infty}\frac{9^{n-1}}{10^n} = \frac{1}{10}\frac{1}{1-(9/10)} = 1$. Thus, m(A) = 0

8. (a) Let F be closed and consider $\mathcal{O}_n = \{x: d(x, F) < 1/n\}$. Then $\mathcal{O}_n$ is open, because the function $f(x) = d(x, F)$ is Lipschitz in the metric d and therefore continuous. Clearly $F \subset \mathcal{O}_n$. Furthermore, $x \in \cap \mathcal{O}_n$ if and only if there is a sequence of elements in F, $\{x_n\}$, such that $x_n \to x$. Since F is closed, we see that $x \in \cap \mathcal{O}_n$ if and only if $x \in F$. Thus $F = \cap \mathcal{O}_n$ and F is a G_δ as desired.

(b) Observe that the set of all rational numbers $\mathbf{Q}$ can be written as a countable union of singleton sets, $\bigcup_{n=1}^{\infty}\{r_n\}$, where the r_n is an ordering of $\mathbf{Q}$ into a sequence. Thus $\mathbf{Q}$ is an F_σ. Note however, that for any countable collection of open sets $\{G_n : \mathbf{Q} \subset G_n\}$, the intersection $G = \cap G_n$ must be, according to Baire Category theorem, an uncountable dense set. Since $\mathbf{Q}$ is countable, we conclude that $\mathbf{Q}$ cannot be represented as a G_δ set.

(c) Let $S = A \cup B$, where $A = [0, 1] \cap \mathbf{Q}$ and $B = [2, 3] - \mathbf{Q}$. Then A is an F_σ that is not a G_δ and B is a G_δ that is not an F_σ (why?). It is immediately clear from the properties of σ algebra that S is a Borel set. However, neither S nor $\mathbf{R} - S$ are F_σ sets: If S were an F_σ set, then for some sequence of closed sets F_n, $S = \bigcup_{n=1}^{\infty} F_n$ and B = $\bigcup_{n=1}^{\infty} F_n \cap [2, 3]$, contradicting the fact that B is not an F_σ set. Similarly, if $\mathbf{R} - S$ is an F_σ, then for some sequence of closed sets C_n, $\mathbf{R} - S = \bigcup_{n=1}^{\infty} C_n$ and, since $[0, 1] - \mathbf{Q} \subset \mathbf{R} - S$, $[0, 1] - \mathbf{Q} = \bigcup_{n=1}^{\infty} C_n \cap [0, 1]$. This is impossible as $[0, 1] - \mathbf{Q}$ is not an F_σ. Thus, neither S nor $\mathbf{R} - S$ are F_σ, which is equivalent to the assertion that S is neither F_σ nor G_δ.

9. (a) Notice first that just like m*, J* is an increasing function: If $A \subset B$, then $J^*(A) \leq J^*(B)$. Hence we immediately have
$$J^*(E) \leq J^*(\overline{E}). \qquad\qquad (1)$$
To get the reverse inequality, observe that if $E \subset \bigcup_{j=1}^{N} I_j$ then $\overline{E} \subset \bigcup_{j=1}^{N} \overline{I}_j$, because the set $\bigcup_{j=1}^{N} \overline{I}_j$ is closed. Consequently,
$$J^*(E) = \inf \sum_{j=1}^{N} |I_j| = \inf \sum_{j=1}^{N} |\overline{I}_j| \geq J^*(\overline{E}). \quad (2)$$
It follows from inequalities (1) and (2) that $J^*(E) = J^*(\overline{E})$ as desired.

(b) Let $E = [0, 1] \cap \mathbf{Q}$. By the work done in part (a), $J^*(E) = J^*(\overline{E}) = J^*([0, 1]) = 1$. While m*(E) = 0.

Remark: $J^*([0, 1]) = 1$, because the sum of the lengths of any finite covering of $[0, 1]$ by open intervals is necessarily bigger than 1 and for any n, $J^*([0, 1]) < \ell\,(-1/2n, 1 + 1/2n) = 1 + 1/n$.

10. (a) Notice that $x \in E_k$ for infinitely many k if and only if $x \in \bigcup\limits_{k=j}^{\infty} E_k$ for every j if and only if $x \in \bigcap\limits_{j=1}^{\infty}\bigcup\limits_{k=j}^{\infty} E_k$. Thus

$$E = \bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_k \ .$$

Since the unions and intersections of measurable sets are again measurable, we conclude that E is measurable.

(b) Assume that $\sum\limits_{k=1}^{\infty} m(E_k) < \infty$. For $\varepsilon > 0$, let j be such that $\sum\limits_{k=j}^{\infty} m(E_k) < \varepsilon$. Therefore

$$m(E) \le m\left(\bigcup_{k=j}^{\infty} E_k\right) \le \sum_{k=j}^{\infty} m(E_k) < \varepsilon .$$

This proves that $m(E) = 0$.

11. Set $E_k^n = \{x \in [0,\ 1] : |\ f_n(x)\ | < k\}$. Then each E_k^n is measurable and for fixed n,

$E_k^n \subset E_{k+1}^n$. Letting $E^n = \bigcup\limits_{k=1}^{\infty} E_k^n = \lim\limits_{k\to\infty} E_k^n$, we get that, since $|\ f_n\ | < \infty$ for a.e. x,

$m([0, 1] - E^n) = 0$.

Since $\lim\limits_{k\to\infty} m(E_k^n) = m(E^n)$, for each n, we can pick an increasing sequence of integers k(n), so that for each n, k(n) is large enough to satisfy

$$m(E^n - E_{k(n)}^n) < 2^{-n} .$$

Define

$$E = \bigcup_{j=1}^{\infty}\bigcap_{n=j}^{\infty} E_{k(n)}^n \ .$$

Then E is measurable and

$$m([0, 1] - E) \le m([0, 1] - \bigcap_{n=j}^{\infty} E_{k(n)}^n) \le \sum_{n=j}^{\infty} m([0,\ 1] - E_{k(n)}^n) = \sum_{n=j}^{\infty} m(E^n - E_{k(n)}^n) \le \sum_{n=j}^{\infty} 2^{-n} \ .$$

Thus, given $\epsilon > 0$, we can choose j large enough so that $\sum\limits_{n=j}^{\infty} 2^{-n} < \epsilon$, which shows that

$m([0, 1] - E) = 0$.

Set $c_n = [k(n)]^2$ and observe that if $x \in E$, $x \in \bigcap\limits_{n=j}^{\infty} E_{k(n)}^n$ for at least one j and therefore, for

all $n \ge j$, $|\ f_n\ | < k(n)$ and

$$\lim_{n\to\infty}\left|\frac{f_n(x)}{c_n}\right| = \lim_{n\to\infty}[k(n)]^{-1}\frac{|f_n(x)|}{k(n)} \leq \lim_{n\to\infty}[k(n)]^{-1} = 0$$

12. Let $A = \{x \in [0, 1]: \lim_{n\to\infty} f_n(x) = 0\}$. Then $m^*([0, 1] - A) = 0$ and A is measurable (since the limit a.e. of measurable functions is again a measurable function). For each $k \in \mathbb{N}$, define

$$E_n^k = \{x \in A: |f_j(x)| < 1/k \text{ for all } j \geq n\}$$

Notice that $E_n^k \subset E_{n+1}^k \nearrow A$. Therefore, we may pick an increasing sequence $n(1) < n(2) < \dots < n(k) < \dots$ such that $m(A - E_{n(k)}^k) < 2^{-k}$ and set

$$E = \bigcup_{r=1}^{\infty}\bigcap_{k=r}^{\infty} E_{n(k)}^k .$$

Then for any r,

$$m([0, 1] - E) = m(A - E) \leq m(A - \bigcap_{k=r}^{\infty} E_{n(k)}^k) \leq \sum_{k=r}^{\infty} 2^{-k} ,$$

implying that $m([0, 1] - E) = 0$.
Define

$$t_n = \begin{cases} 0 & \text{if } n \neq n(k) \\ 1/k & \text{if } n = n(k) \end{cases}.$$

Then $\sum_{n=1}^{\infty} t_n = \sum_{k=1}^{\infty}\frac{1}{k} = \infty$. However, for $x \in E$, there is some r such that $x \in \bigcap_{k=r}^{\infty} E_{n(k)}^k$. Thus, $|f_{n(k)}(x)| < 1/k$ for all $k \geq r$ and, consequently,

$$\sum_{n=1}^{\infty} t_n |f_n(x)| = \sum_{k=1}^{\infty}\frac{|f_{n(k)}(x)|}{k} = \sum_{k=1}^{r-1}\frac{|f_{n(k)}(x)|}{k} + \sum_{k=r}^{\infty}\frac{|f_{n(k)}(x)|}{k} \leq \sum_{k=1}^{r-1}\frac{|f_{n(k)}(x)|}{k} + \sum_{k=r}^{\infty}\frac{1}{k^2} < \infty.$$

13. For any measurable function f on $\mathbb{R}^d$ there is a sequence of step functions $\{\psi_n\}_{n=1}^{\infty}$ such that $\psi_n \to f$ a.e. x. Therefore, to show that f is the limit of a sequence of continuous functions a.e., it suffices to establish this in the spatial case when f is a step function. Since every step function $\psi = \sum_{j=1}^{N} a_j \chi_{R_j}$ is a finite linear combination of characteristic functions over rectangles, we may assume further without any loss of generality that $f = \chi_R$ where $R = [a_1, b_1] \times \dots \times [a_d, b_d]$.
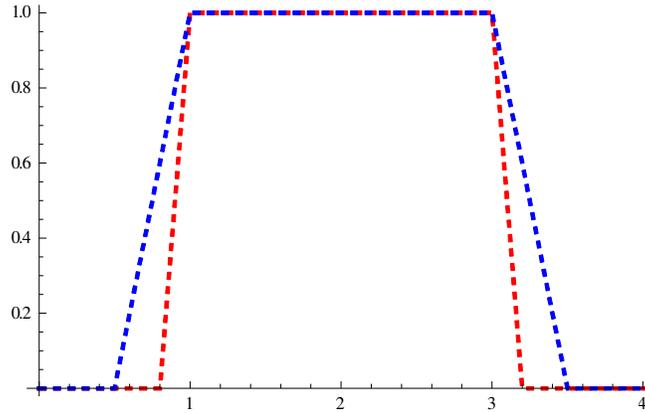Notice that

$$\chi_R = \chi_{[a_1, b_1]} \cdot \dots \cdot \chi_{[a_d, b_d]}$$

and for each $1 \leq j \leq d$, we may define a sequence of continuous functions $F_n^j : \mathbb{R} \to \mathbb{R}$ that are given by

$$F_n^j(x) = \begin{cases} 0 & \text{if } x \notin (a_j - 1/n, \ b_j + 1/n) \\ 1 & \text{if } x \in (a_j, \ b_j) \\ piecewise \ linear & otherwise \end{cases}$$

Below are the plots of F_2^j (in blue) and F_5^j (in red) in the special instance when $a_j = 1$ and $b_j = 3$:

Clearly,

$$F_n^j \to \chi_{[a_j, \ b_j]} \text{ (pointwise)}.$$

Define $f_n : \mathbf{R}^d \to \mathbf{R}$ by

$$f_n(x_1, \ldots, x_d) = F_n^1(x_1) \cdot \ldots \cdot F_n^d(x_d).$$

Then

$$\lim_{n \to \infty} f_n = f$$

as desired.

14. (a) Assume that A is open. Then for any $x \in A$, there is some $\epsilon > 0$ such that
$$B_\varepsilon(x) \subset A.$$

For any $y \in B$, the open ball
$$B_\varepsilon(x + y) = y + B_\varepsilon(x) \subset A + B.$$

This shows that A + B is open.

(b) Assume now that A and B are both closed. Then $A = \bigcup_{k=1}^{\infty} A_k$ and $B = \bigcup_{j=1}^{\infty} B_j$,

where the A_k and B_j are compact sequences of sets. It follows that
$$A + B = \bigcup_{k,j} A_k + B_j$$

Where the union is taken over all combinations of k and j.

Notice that each set $A_k + B_j$ must be compact: Let $x_n + y_n$ be a sequence in $A_k + B_j$. Since A_k is compact, there is a subsequence $x_{n(p)}$ of x_n such that $x_{n(p)} \to x \in A_k$. Since B_j is

compact, there is a subsequence $y_{n(p(r))}$ of $y_{n(p)}$ such that $y_{n(p(r))} \to y \in B_j$. Therefore the sequence $x_n + y_n$ has $x_{n(p(r))} + y_{n(p(r))}$ as a convergent subsequence, which converges in $A_k + B_j$.

It follows that A + B is the countable union of compact sets and is therefore an F_σ.

(c) Counterexample in **R**:

Let A = {-n: n $\in$ **N**} and B = {n + 1/n: n $\in$ **N**}. Then A and B are closed. However, A + B is not closed as it contains the sequence 1/n, but doesn't contain 0.

Counterexample in $\mathbf{R}^2$:

Let A = {(n, 1−1/n): n $\in$ **N**} and B = {(x, 0): x $\in$ **R**}. Then A and B are closed, whereas A + B isn't; Every point in the set {(x, 1): x $\in$ **R**} is a limit point of A + B that is not contained in A + B.

15. (a) Observe that the cantor set Δ is the set of all x $\in$ [0, 1] that have a ternary representation that uses only 0 and 2. That is, $x = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ where $a_n = 0$ or 1. Any

z $\in$ [0, 1] can be represented as a sum of two elements, z = x + y, where x has a ternary representation that only uses 0 and 2 and y has a ternary representation that uses only 0 and 1. For example, 0.010211 (mod 3) = 0.000200 + 0.010011 (mod 3). Cleary y $\in$ Δ/2 and [0, 1] $\subset$ Δ + Δ/2. Hence m(Δ + Δ/2) $\geq$ m([0, 1]) = 1. Since m(Δ) = 0, m(Δ/2) = ½ m(Δ) = 0.

(b) Let A = [0, 1] ×{0} and B = {0} ×[0, 1], then A + B = [0, 1] × [0, 1]. In $\mathbf{R}^2$, m(A +B) = 1, since A + B is a unit square. m(A) = m(B) = 0, however, because for any $\epsilon >$ 0 the set A, for instance, can be contained in the rectangular strip [0, 1] ×[-ϵ, ϵ]. Since the $\mathbf{R}^2$ measure of a rectangle is its area, we see that m(A) < m([0, 1] ×[-ϵ, ϵ]) = 2ϵ.

16. Suppose that m*(A) > 0, where A $\subset$ **R**. Then A = $\bigcup_{n=-\infty}^{\infty} A \cap [n, \ n+1]$ and m*(A) $\leq$

$\sum_{n=-\infty}^{\infty} m*(A \cap [n, \ n+1])$ and it follows that for at least one n, m*(A $\cap$ [n, n + 1]) > 0.

We will therefore attempt to construct a nonmeasurable subset of A $\cap$ [n, n + 1]: For any $\alpha \in$ A $\cap$ [n, n + 1], let ξ_α = {x $\in$ A $\cap$ [n, n + 1]: x$-\alpha \in$ **Q**}. Then for $\alpha, \beta \in$ A $\cap$ [n, n + 1],

$$\xi_\alpha \cap \xi_\beta = \emptyset \text{ if } \alpha \neq \beta$$

and

$$\xi_\alpha = \xi_\beta \text{ otherwise.}$$

Pick exactly one representative α from each ξ_α and set N to be the collection of these representatives. Observe that for any x, y $\in$ [n, n + 1], we have $|x -y| \leq 1$. Consequently,

$A \cap [n, n + 1] \subset \bigcup\limits_{n=1}^{\infty}(r_n + N) \subset$ [n–1, n + 2], where r_n enumerates all the rational numbers in [-1, 1]. Notice also that for n ≠ m, the sets r_n + N and r_m + N are disjoint, because for any

$\alpha, \beta \in N$, $r_n + \alpha = r_m + \beta$ if and only if $\xi_\alpha = \xi_\beta$ which, by the construction of N, can only happen if and only if $\alpha = \beta$, $r_n = r_m$, and therefore n = m.

Thus the set N cannot be measurable, for otherwise we have

$$0 < m^*(A \cap [n, n + 1]) \le m\left(\bigcup\limits_{n=1}^{\infty}(r_n + N)\right) = \sum\limits_{n=1}^{\infty}m(r_n + N) = \sum\limits_{n=1}^{\infty}m(N) \le m([n-1, n + 2]) = 3$$

which is impossible.

17. Let c: $\Delta \to$ [0, 1] be the Cantor function defined by $c(x) = \sum\limits_{n=1}^{\infty}\frac{a_n}{2^n}$ when x = $\sum\limits_{n=1}^{\infty}\frac{2a_n}{3^n}$

($a_n = 1$ or 2) is an element of the Cantor set represented in ternary decimal expansion.

Then c is an increasing function on Δ that is onto [0, 1]. This function can be extended to an increasing continuous onto function g: [0, 1] → [0, 1] which is given by

g(x) = sup c(y), where the sup is taken over all y ≤ x. We construct a strictly increasing, onto, continuous function f: [0, 1] → [0, 2] by setting f(x) = x + g(x).

Since f and g are continuous, they map compacts sets to compact sets. Hence, because the Cantor set Δ is compact, both f(Δ) and g(Δ) are compact and therefore measurable and of finite measure. Recall that g(Δ) = [0, 1] and therefore m (g(Δ)) = 1. We claim m(f(Δ)) ≥ m(g(Δ)). To see this, for ϵ > 0, let $\mathcal{O} \supset$ f(Δ) be an open set such that m($\mathcal{O}$) <

m(f(Δ)) + ϵ. Then $\mathcal{O} = \bigcup\limits_{n=1}^{\infty}I_n$ is a countable union of disjoint open intervals. For each I_n,

pick $x_n \in \Delta$ such that f(x_n) $\in I_n$ and set

$$J_n = I_n - x_n = \{y - x_n : y \in I_n\}.$$

Then the open set $\mathcal{O}' = \bigcup\limits_{n=1}^{\infty}J_n$ contains [0, 1] = g(Δ). To verify this, select any x $\in \Delta$ and

since $\mathcal{O} \supset$ f(Δ), we must have f(x) $\in I_n$ for some n. This implies that g(x) = f(x)$-$x $\in J_n$ as verified by analyzing the following two cases:

  Case 1: Suppose x ≤ x_n. Then [f(x), f(x_n)]$\subset I_n$ and

$$[g(x), g(x_n)] = [f(x)-x, f(x_n)-x_n] \subset [f(x)-x_n, f(x_n)-x_n] \subset J_n.$$

In particular, g(x) $\in J_n$.

  Case 2: Suppose x ≥ x_n. Then [f(x_n), f(x)]$\subset I_n$ and

$$[g(x_n), g(x)] = [f(x_n)-x_n, f(x)-x] \subset [f(x_n)-x_n, f(x)-x_n] \subset J_n.$$

In particular, g(x) $\in J_n$.

It follows that

$$m(g(\Delta)) \le m(\mathcal{O}') \le \sum_{n=1}^{\infty} m(J_n) = \sum_{n=1}^{\infty} m(I_n) = m(\mathcal{O}) < m(f(\Delta)) + \epsilon.$$

Hence

$$m(f(\Delta)) \ge m(g(\Delta)) = m([0, 1]) = 1.$$

By the previous exercise, we may pick a nonmeasurable subset $N \subset f(\Delta)$ and set $A = f^{-1}(N) \subset \Delta$. Then A has measure 0 and is therefore measurable, whereas $f(A) = N$ is not. Thus a continuous function may map a measurable set to a nonmeasurable set even with the added hypothesis that the function is strictly increasing.

18. Consider the function f: $[0, 1] \to [0, 2]$ from the previous exercise. Notice that any continuous function on a compact set is a closed map. That is, f maps closed subsets of $[0, 1]$ to closed subsets of $[0, 2]$. Since f is strictly increasing and onto, f is invertible with a continuous inverse $f^{-1}: [0, 2] \to [0, 1]$ (If $F \subset [0, 1]$ is closed, then $(f^{-1})^{-1}(F) = f(F)$ is a closed subset of $[0, 2]$. Thus, the inverse image of a closed set under f^{-1} is closed, which means that f^{-1} must be continuous.) In particular, f is a homeomorphism; f maps closed sets to closed sets and open sets to open sets.

Because f is bijective, for any sequence of subsets $H_n \subset [0, 1]$, we have

$$f\left(\bigcup_{n=1}^{\infty} H_n\right) = \bigcup_{n=1}^{\infty} f(H_n) \text{ and } f\left(\bigcap_{n=1}^{\infty} H_n\right) = \bigcap_{n=1}^{\infty} f(H_n). \qquad (1)$$

Thus, f maps an F_σ to an F_σ-set and a G_δ to a G_δ-set. Moreover, a closer investigation of the identities (1) reveals that f maps any Borel set to a set of the same type. For instance, if B is an $F_{\sigma\delta}$-set, B = $\bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} F_k^j$, where the F_k^j are closed. Then

$$f(B) = \bigcap_{j=1}^{\infty} f\left(\bigcup_{k=1}^{\infty} F_k^j\right) = \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} f(F_k^j),$$

where f(F_k^j) is closed. Hence f(B) is also an $F_{\sigma\delta}$-set.

Let $A \subset \Delta$ be as in the previous exercise. Then A is measurable, while $f(A) = N$ isn't. But if A were a Borel set, f(A) would have been a Borel set as well, which is impossible as all Borel sets are measurable.

19. Observe that there is a one-to-one correspondence between the set $\mathbf{N}^2 = \{n^2 : n \in \mathbf{N}\}$ and $\mathbf{R} - [0, 1] \cap \mathbf{Q}$ and a one-to-one correspondence between the set of square free integers $\mathbf{N} - \mathbf{N}^2$ and $[0, 1] \cap \mathbf{Q}$. Thus, $\mathbf{Q}$ can be ordered into a sequence $\{r_n\}_{n=1}^{\infty}$ such that $r_n \in [0, 1] \cap \mathbf{Q}$ whenever $n \in \mathbf{N} - \mathbf{N}^2$ and $r_n \in \mathbf{R} - [0, 1] \cap \mathbf{Q}$ when $n \in \mathbf{N}^2$. In this case

$$m\left(\bigcup_{n=1}^{\infty}\left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n}\right)\right) = m\left(\bigcup_{n \in N-N^2}\left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n}\right) \bigcup \bigcup_{n \in N^2}\left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n}\right)\right).$$

Since 2 is the smallest number in $\mathbf{N} - \mathbf{N}^2$,

$$\bigcup_{n \in N-N^2} \left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n} \right) \subset (\text{-}1/2, \ 1 + 1/2) \subset (\text{-}1, \ 2).$$

Therefore,

$$m\left(\bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n} \right) \right) \leq m\left(\bigcup_{n \in N-N^2} \left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n} \right) \right) + m\left(\bigcup_{n \in N^2} \left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n} \right) \right),$$

which implies

$$m\left(\bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n} \right) \right) \leq 3 + \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty .$$

The compliment of $\bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{n}, \ r_n + \frac{1}{n} \right)$ has, consequently, an infinite measure and

therefore must contain an uncountably infinite collection of real numbers.

20. Notice that $E = A \cup E-A$ where $E-A \subset B-A$. Since A and B are measurable sets of finite measure satisfying $A \subset B$ and $m(A) = m(B)$, we have
$$m^*(E-A) \leq m(B-A) = m(B) - m(A) = 0.$$
Thus, upon recalling that all sets of zero measure are measurable, we see that E is the union of two measurable sets. Hence E is measurable.

21. Let $B = A - \mathbf{Q}$. Then $m(B) = m(A)$. Observe that $B = \bigcup_{n=1}^{\infty} B_n$ where $B_n = B \cap [\text{-}n, n]$ is an

increasing sequence of bounded measurable sets with $B_n \nearrow B$. Thus, there exists an n
large enough so that $x < m(B_n) < m(B)$. And because B_n is measurable,
for $\epsilon = m(B_n) - x$, there is a closed set $C \subset B_n$ such that $m(B_n - C) = m(B_n)-m(C) < \epsilon$.
Hence $m(C) > x$. Note that C is closed and bounded and therefore compact. Let $a = \inf C$
and $b = \sup C$. Also, C is a subset of B and therefore contains no rational numbers.
Define $f(t) = m((\text{-}\infty, t] \cap C)$. Then f is Lipschitz:
Observe that for $t < s$, $f(t) \leq f(s)$ and
$$f(s) = m(\{(\text{-}\infty, t] \cup (t, s]\} \cap C) = f(t) + m((t, s] \cap C) \leq f(t) + (s-t).$$
Thus,
$$| f(s)-f(t)| \ = f(s)-f(t) \leq s-t = \ |s-t|.$$
In particular, f is continuous.
Let $a = \inf C$ and $b = \sup C$. Since C is bounded, a and b are finite numbers for which
$f(a) = 0$ and $f(b) = m(C) > x$. By the intermediate value property, there is some number
$a < t_0 < b$ such that $f(t_0) = x$. Then $F = (\text{-}\infty, t_0] \cap C$ is the desired set.

22. We may assume that E is a bounded set, because $m^*(E) > 0$ implies that
$m^*(E \cap [\text{-}n, n]) > 0$ for all large enough n. Thus, without loss of generality, $m^*(E) < \infty$.
Recall that $m^*(E) = \inf m(\vartheta)$ where the infimum is taken over all open sets $\vartheta \supset E$. For

$0 < \alpha < 1$, pick $\epsilon > 0$ that satisfies $\alpha = 1/(1 + \epsilon)$ and along with it pick an open set $\vartheta \supset E$, such that

$$m(\vartheta) < (1 + \epsilon) \, m^*(E). \tag{1}$$

Notice that inequality (1) is equivalent to $\alpha m(\vartheta) < m^*(E)$. Upon expressing $\vartheta = \bigcup_{n=1}^{\infty} I_n$ as the union of disjoint open intervals, we see that

$$\sum_{n=1}^{\infty} m(\alpha \, I_n) = \alpha m(\vartheta) < m^*(E) \le \sum_{n=1}^{\infty} m^*(E \cap I_n) \tag{2}$$

Since the series on the right side of (2) is bigger than the left series, it follows that for at least one n, $m(\alpha \, I_n) \le m^*(E \cap I_n)$.

23. By exercise 22, we can find an open interval I of finite length such that $m(E \cap I) > \alpha \, m(I)$, where $0 < \alpha < 1$. Under the appropriate choice of α, we will show that E – E contains the interval J = (-m(I)/2, m(I)/2). Now $E \cap I - E \cap I$ contains J if and only if for each $x \in J$, the sets $x + E \cap I$ and $E \cap I$ are not disjoint. We prove that $(x + E \cap I) \cap (E \cap I) \ne \emptyset$ by establishing

$$m((x + E \cap I) \cup (E \cap I)) < m(x + E \cap I) + m(E \cap I) = 2m(E \cap I)$$

for all $x \in J$.
We have

$$m((x + E \cap I) \cup (E \cap I)) \le m((x + I) \cup I) \le m((\sup(J) + I) \cup I) = (3/2)m(I) < 2m(I).$$

Where, in the inequality above we note that $m((x + I) \cup I)$ is maximized when $x + I$ is shifted away from I as far as possible. The largest shift is smaller than $x = m(I)/2$ and therefore $x + I$ and I must have an open interval whose length is not smaller than $m(I)/2$. In particular $x + I$ and I are never disjoint.
Finally,

$$m((x + E \cap I) \cup (E \cap I)) \le (3/2)m(I) < (3/2) \, (1/\alpha) \, m(E \cap I) < 2 \, m(E \cap I)$$

whenever $(3/2) \, (1/\alpha) < 2$ or equivalently, when $\alpha > \frac{3}{4}$.

24. We will reduce the proof to the special case E – E treated in the previous exercise with the help of the following observations:

Observation 1: By setting K = -F = {-x: x ∈ F}, E + F becomes E – K. Thus, without loss of generality, we only need to prove that E – F contains an interval.

Observation 2: If $r \in \mathbf{R}$, (r + E) – F = r + (E – F). Hence, E – F contains the interval I if and only if (r + E) – F contains the interval r + I.

Observation 3: For each $0 < \alpha < 1$, there exist open intervals I and J with m(I) = m(J) such that $m(E \cap I) > \alpha \, m(I)$ and $m(F \cap J) > \alpha \, m(J)$. To see this, first notice that every open set $\vartheta \subset \mathbf{R}^d$ can be written as a countable union of almost disjoint closed cubes. Furthermore, we may construct these cubes so that the side length of each cube is some integer power of 2. Thus, if U is an open set containing E and satisfying $\alpha \, m(U) < m(E)$, we can repeat the argument in exercise 22 to find (this time closed) interval I_1 with $m(I_1) = 2^n$ for some $n \in \mathbf{Z}$ so that $\alpha \, m(I_1) < m(E \cap I_1)$. Similarly, we can find a closed

interval J_1 with $m(J_1) = 2^m$ for some $m \in \mathbf{Z}$ so that $\alpha \, m(J_1) < m(F \cap J_1)$. Without loss of generality, $m(J_1) \le m(I_1)$ and it follows by construction that $m(I_1) = k \, m(J_1)$, where k is a positive integer. In particular, I_1 is the union of k almost disjoint closed intervals $I_1^1, \ldots, I_1^k$ with $m(I_1^j) = m(J_1)$ for each j. Finally, since

$$\alpha \, m(I_1) = \sum_{j=1}^{k} \alpha \, m(I_1^j) < m(E \cap I_1) = \sum_{j=1}^{k} m(E \cap I_1^j),$$

it follows that for at least one j, $\alpha \, m(I_1^j) < m(E \cap I_1^j)$. The intervals $I = \mathrm{interior}(I_1^j)$ and $J = \mathrm{interior}(J_1)$ establish our claim.

We proceed to show that $m((r + E) \cap F) > 0$ for some r. It will then follow that $(r + E) - F$ must contain a set of the form $K - K$, where $K = (r + E) \cap F$ is a set of positive measure. Pick open intervals I and J obtained in observation 3. Shift the interval I by an appropriate r so that $r + I = J$. Then, for an appropriately chosen value of α (to be specified below),

$$m([(r + E) \cap J] \cap [F \cap J]) > 0.$$

Otherwise,

$$m(J) \ge m([(r + E) \cap J] \cup [F \cap J]) = m((r + E) \cap J) + m(F \cap J) > 2\alpha \, m(J).$$

But $m(J) > 2\alpha \, m(J)$ if and only if $\alpha < \tfrac{1}{2}$. Thus $\tfrac{1}{2} \le \alpha < 1$ guarantees $m((r + E) \cap F) > 0$ as desired.

25. Let $r = 0$ and $E = \bigcup_{2k=0}^{\infty} E_{2k}$, where $E_{2k} = I_{2k} \cup (-I_{2k})$ and $I_{2k} = [2^{-2k-1}, \, 2^{-2k}]$. Set

$h_n = \sum_{k=n}^{\infty} 2^k = 2^{-n+1}$. We show that

$$\lim_{n \to \infty} \frac{m(E \cap [-h_{2n}, \, h_{2n}])}{2h_{2n}} \ne \lim_{n \to \infty} \frac{m(E \cap [-h_{2n+1}, \, h_{2n+1}])}{2h_{2n+1}}.$$

Now,

$$\lim_{n \to \infty} \frac{m(E \cap [-h_{2n}, \, h_{2n}])}{2h_{2n}} = \lim_{n \to \infty} \frac{\sum_{k=n}^{\infty} m(I_{2k})}{2^{-2n+1}} = \lim_{n \to \infty} \frac{\sum_{k=n}^{\infty} 2^{-2k-1}}{2^{-2n+1}} = \frac{1}{3},$$

Whereas

$$\lim_{n \to \infty} \frac{m(E \cap [-h_{2n+1}, \, h_{2n+1}])}{2h_{2n+1}} = \lim_{n \to \infty} \frac{\sum_{k=n}^{\infty} m(I_{2k+2})}{2^{-2n}} = \lim_{n \to \infty} \frac{\sum_{k=n}^{\infty} 2^{-2k-3}}{2^{-2n}} = \frac{1}{6}$$

and we are done.